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An elastoplastic theory of dislocations as a physical field theory with torsion

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Abstract

We consider a static theory of dislocations with moment stress in an anisotropic or isotropic elastoplastic material as a $T(3)$ gauge theory. We obtain Yang–Mills-type field equations which express the force and the moment equilibrium. Additionally, we discuss several constitutive laws between the dislocation density and the moment stress. For a straight screw dislocation, we find the stress field which is modified near the dislocation core due to the appearance of moment stress. For the first time, we calculate the localized moment stress, the Nye tensor, the elastoplastic energy and the modified Peach–Koehler force of a screw dislocation in this framework. Moreover, we discuss the straightforward analogy between a screw dislocation and a magnetic vortex. The dislocation theory in solids is also considered as a three-dimensional effective theory of gravity.

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1. Introduction

Defects in crystals, e.g. elementary point defects, dislocations and stacking faults, play a fundamental role in determining the behaviour and properties of crystalline materials. In principle, point defects make the crystal viscoelastic, whereas dislocations cause plasticity. After plastic bending or twisting a crystal contains dislocations which give rise to a lattice curvature. The dislocations can be directly observed by the help of high resolution electron microscopes. The crystallographic or topological defects not only influence the mechanical, but also the electrical, magnetic and other properties. All these defects break the symmetry of the ideal crystal (defect-free crystal), as an analogue of a trivial vacuum, to the real crystal as a nontrivial vacuum.

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The traditional description of elastic fields produced by dislocations is based on the classical theory of linear elasticity. This approach works quite well for the strain and stress fields far from the core. However, the components of these fields are singular at the dislocation line and this theory, often applied to practical problems, misses the important feature of plasticity. On the other hand, in conventional plasticity theories no internal length scale enters the constitutive law and no size effects are predicted.

Therefore, it is quite natural to think of dislocation theory as a theory of elastoplasticity. In this framework, it is possible to define a characteristic internal length by the help of a new material constant. In analogy to the theory of elementary particle physics and gravity, we propose a (static) elastoplastic field theory of dislocations (see also [1]).

A gauge theory of dislocations is formally given in [2–4] but without considering the moment stress. Moreover, their gauge Lagrangian of dislocation is not the most general one for an isotropic material because it contains only one material constant and has been chosen in a very special form to be quadratic in translational gauge field strength. Recently, Malyshev [5] discussed that the gauge Lagrangian used by Edelen *et al* does not lead to the correct solution for an edge dislocation within a linear approximation. Additionally, any correct gauge theory of dislocations must give the well-established results obtained earlier with the older theory of dislocations. For instance, Kadić and Edelen [2, 3] and Edelen and Lagoudas [4] find in their gauge theory of dislocations that the far field stress of a screw dislocation decays exponentially and the near field decay is found to go with r^{-1} . Obviously, this is an important difference between the gauge theory of dislocations and classical dislocation theory. Fortunately, about ten years later Edelen [6] realized that the solutions for a screw and an edge dislocation given in [2–4] are unphysical.

The aim of this paper is to develop a static theory of dislocations which makes use of the concepts of field strength, excitation and constitutive functions. Like Maxwell's field equations, the theory of dislocations consists of two sets of field equations which are connected by constitutive laws. This theory is a kind of axiomatic field theory of dislocations similar to axiomatic Maxwell's theory, which has recently been given by Hehl *et al* [7, 8]. Additionally, this dislocation theory is a three-dimensional translation gauge theory [1] which makes use of the framework of metric affine gauge (MAG) theory given by Hehl *et al* [9–12]. We discuss in detail as a physical example the elastoplastic properties of a screw dislocation as a crystal defect. We show that the solution of the gauge field equations for a screw dislocation in an infinite medium gives the classical far field and a modified near field.

Moreover, we discuss a straightforward analogy between a screw dislocation and a magnetic vortex in type-II superconductors [13, 14]. Additionally, we discuss the dislocation theory as a three-dimensional theory of gravity.

As a formalism we use the calculus of exterior differential forms: for our conventions see [11, 15–17].

2. Elasticity theory

In classical elasticity theory (see [17]) the material body is identified with a three-dimensional manifold \mathcal{M}^3 which is embedded in the three-dimensional Euclidean space \mathbb{R}^3 . We distinguish between the material or the final coordinates of \mathcal{M}^3 , $a, b, c, \dots = 1, 2, 3$, and the (holonomic) Cartesian coordinates of the reference system (defect-free or ideal reference system) \mathbb{R}^3 , $i, j, k, \dots = 1, 2, 3$. A *deformation* of \mathbb{R}^3 is a mapping $\xi : \mathbb{R}^3 \rightarrow \mathcal{M}^3$. This deformation or distortion 1-form is defined by

$$\vartheta^a = B_i^a dx^i = d\xi^a \quad (2.1)$$

and can be identified with the soldering form and the (orthonormal) coframe, respectively. Here d denotes the three-dimensional exterior derivative. Since

$$d\vartheta^a = dd\xi^a = 0 \tag{2.2}$$

the elastic distortion (2.1) is compatible or holonomic and the body manifold is simply connected. The compatible distortion 1-form (2.1) is invariant under ‘rigid’, i.e. constant translational, transformations:

$$\xi^a \longrightarrow \xi^a + \tau^a \tag{2.3}$$

where τ^a are constant translations.

Using the orthonormal coframe, the volume 3-form is defined by

$$\eta := \frac{1}{3!} \epsilon_{abc} \vartheta^a \wedge \vartheta^b \wedge \vartheta^c = \frac{1}{3!} B \epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k \tag{2.4}$$

with $B \equiv \det(B_i^a)$ and ϵ_{abc} is the Levi-Civita symbol, $\eta_a := e_a \rfloor \eta$, $\eta_{ab} := e_a \rfloor e_b \rfloor \eta$, and $\eta_{abc} := e_a \rfloor e_b \rfloor e_c \rfloor \eta$. Here \rfloor denotes the interior product with

$$e_a \rfloor \vartheta^b = B_a^i B_i^b = \delta_a^b \quad e_a = B_a^i \partial_i \tag{2.5}$$

and \wedge the exterior product ($A \wedge B = A \otimes B - B \otimes A$). In the following we use the Hodge duality operation $*$. For a p -form α , the dual $(3 - p)$ -form ($p \leq 3$) with $**\alpha = \alpha$ is given by

$$*\alpha = \frac{1}{p!} (\alpha^{a_1 \dots a_p} e_{a_p} \wedge \dots \wedge e_{a_1}) \rfloor \eta. \tag{2.6}$$

The *Cauchy–Green strain tensor* G is defined as the metric of the final state:

$$G = \delta_{ab} \vartheta^a \otimes \vartheta^b = \delta_{ab} B_i^a B_j^b dx^i \otimes dx^j = g_{ij} dx^i \otimes dx^j \tag{2.7}$$

where $\delta_{ab} = \text{diag}(+++)$. We can interpret the strain tensor as a kind of effective field which is formed during the deformation. For an incompressible material, it holds the following condition (constraint of incompressibility):

$$\det(B_i^a) = \sqrt{\det(g_{ij})} = 1. \tag{2.8}$$

Finally, the relative strain tensor (Green–Lagrange strain tensor) E is given by

$$2E = G - 1 = (g_{ij} - \delta_{ij}) dx^i \otimes dx^j. \tag{2.9}$$

It measures the change of the metric between the undeformed and the deformed state.

Let us now consider the elastic strain Lagrangian. For simplicity we assume a linear *constitutive law*. The elastic (anisotropic) Lagrangian is given in terms of the potential (strain) energy

$$\mathcal{L}_{\text{strain}} = -W\eta. \tag{2.10}$$

The *potential energy* is given by

$$W = \frac{1}{2 \cdot 4!} (C \rfloor E \otimes E) = \frac{1}{2} C^{ijkl} E_{ij} E_{kl} \tag{2.11}$$

where the elasticity tensor [17], which describes the elastic properties of the material under consideration, is defined by

$$C = C^{ijkl} \partial_i \otimes \partial_j \otimes \partial_k \otimes \partial_l \quad C^{ijkl} = C^{jikl} = C^{ijlk} = C^{klij}. \tag{2.12}$$

The *elastic force stress* is the elastic response quantity pertaining to the distortion and is defined by

$$\Sigma_a := \frac{\delta \mathcal{L}_{\text{strain}}}{\delta \vartheta^a}. \tag{2.13}$$

We speak of force stresses to distinguish them from the so-called moment stress (transmission of moments). In particular, Σ_a is a \mathbb{R}^3 -valued odd (or axial) differential form. Here Σ_a is given by (see also [5, 18])

$$\Sigma_a = -\sigma_a^l \eta_l - W \eta_a \quad (2.14)$$

with $\sigma^{kl} := \partial W / \partial E_{kl} = C^{ijkl} E_{ij}$ and $\sigma_a^l = \sigma^{kl} \delta_{ac} B_k^c$ is the first Piola–Kirchhoff stress tensor. The second term in Σ_a is due to the variation of the volume 3-form η . In this way, equation (2.14) corresponds to Eshelby's elastic stress tensor [19, 20]. It is the stress tensor for a compressible medium and appears in quite a natural way in this framework.

The *elastic strain energy* density $\mathcal{E}_{\text{strain}}$ is defined as the Hamiltonian

$$\mathcal{E}_{\text{strain}} := -\mathcal{L}_{\text{strain}} = W \eta. \quad (2.15)$$

3. Elastoplasticity— $T(3)$ gauge theory of dislocations

In this section, we discuss the theory of elastoplasticity as a translational gauge theory ($T(3)$ gauge theory). We postulate a local $T(3)$ invariance for the field ξ^a :

$$\xi^a \longrightarrow \xi^a + \tau^a(x) \quad (3.1)$$

where $\tau^a(x)$ are local translations. If we do it, the invariance of the compatible distortion (2.1) is lost under the local transformations. In order to kill the invariance violating terms, we have to introduce a compensating gauge potential 1-form ϕ^a , which transforms under the local transformations in a suitable form:

$$\phi^a \longrightarrow \phi^a - d\tau^a(x). \quad (3.2)$$

The field ϕ^a couples in a well determined way to the field ξ^a :

$$\vartheta^a = d\xi^a + \phi^a \quad (3.3)$$

such that the distortion 1-form (3.3), which is now incompatible, is invariant under local $T(3)$ transformations. The coupling in (3.3) between the translational gauge potential ϕ^a and the vector field ξ^a is a kind of translational covariant derivative acting on ξ^a (see also [1, 11]). Accordingly, the incompatible distortion (3.3) can be understood as the (minimal) replacement of the compatible distortion (2.1) in $T(3)$ gauge theory

$$d\xi^a \longrightarrow d\xi^a + \phi^a. \quad (3.4)$$

The minimal coupling argument leads to the substitution in the strain energy

$$W(d\xi^a) \longrightarrow W(d\xi^a, \phi^a). \quad (3.5)$$

A translational gauge theory is thus a theory which corresponds to the gauge invariance with respect to local displacement transformations.

The reason for plasticity are dislocations and the material gives rise to an elastic response. The distortion or soldering form (3.3) is now anholonomic due to $d\vartheta^a \neq 0$ and the incompatible part is caused by defects. The presence of dislocations makes the final crystallographic coordinate system anholonomic and the body manifold after an incompatible deformation is not simply connected.

If we interpret the dislocation gauge potential ϕ^a as the negative *plastic distortion*, we observe

$$d\vartheta^a = -d\phi^a. \quad (3.6)$$

Finally, the *total distortion* contains elastic and plastic contributions according to

$$d\xi^a = \vartheta^a - \phi^a \quad (3.7)$$

so that the total distortion is compatible and can be written in terms of the mapping function ξ^a .

In [1] we have seen that the even (or polar) 1-form ϕ^a in equation (3.3) can be interpreted as the translational part of the generalized affine connection in a Weitzenböck space when the linear connection ω^a_b is globally gauged to zero. Such a space carries torsion, but no curvature.

On the other hand, a dislocation is a translational defect which causes the deviation from the euclidicity of the crystal, sometimes called the inner geometry. This inner geometry of a crystal with dislocations can be described as a space with teleparallelism, i.e. a flat space with torsion [21–24]. In this context, a clear physical interpretation of torsion was discovered for the first time. The differential geometric notion of torsion was originally introduced by Cartan [25]. In this frame, Cartan had already found that torsion is related to translations. Additionally, we can identify the vector-valued 0-form ξ^a as Trautman’s *generalized Higgs field* [26]. In contrast to gravity, where the physical meaning of ξ^a is not completely clarified (see [11] and the references given there), we cannot use the ‘gauge’ condition $\xi^a = x^a$ or $d\xi^a = 0$ in our translation gauge theory of dislocations. Consequently, the translational part of the generalized affine connection cannot be identified with the soldering form. In other words, the translational gauge theory of dislocations is a theory where the torsion and the translational part of the generalized affine connection play a physical role.

Now we define the translational field strength (torsion) 2-form T^a , in the gauge $\omega^a_b \equiv 0$, as²

$$T^a = d\vartheta^a = d\phi^a = \frac{1}{2} T^a_{ij} dx^i \wedge dx^j. \tag{3.8}$$

One obtains the conventional dislocation density tensor α^a_i from T^a by means of $\alpha^a_i = \frac{1}{2} \epsilon_i^{jk} T^a_{jk}$. Here, T^a is an even (or polar) 2-form with values in \mathbb{R}^3 . By taking the exterior derivative one gets the translational Bianchi identity

$$dT^a = 0. \tag{3.9}$$

Physically, equation (3.9) means that dislocations cannot end inside the body [27]. A characteristic quantity which expresses a fundamental property of dislocations is the Burgers vector. The Burgers vector³ is defined by integrating around a closed path γ (Burgers circuit) encircling a dislocation

$$b^a = \oint_{\gamma} \vartheta^a = \int_S T^a \tag{3.10}$$

where S is any smooth surface with boundary $\gamma = \partial S$. Thus, the dislocation shows itself by a closure failure (Burgers vector), i.e. a translational misfit. This means that the closed parallelogram of the ideal crystal does not close in the dislocated crystal. For a distribution of dislocations we have to interpret b^a in equation (3.10) as the sum of the Burgers vectors of all dislocations which pierce through the surface S (dislocation density flux through the surface S).

To complete the field theory of dislocations, we have to define the excitation with respect to the dislocation density. We make the most general Yang–Mills-type ansatz

$$\mathcal{L}_{\text{dist}} = -\frac{1}{2} T^a \wedge H_a. \tag{3.11}$$

Here the *moment stress* 1-form H_a is defined by (see [1, 18, 28–31])

$$H_a := -\frac{\partial \mathcal{L}_{\text{dist}}}{\partial T^a} \tag{3.12}$$

as an odd (or axial) \mathbb{R}^3 -valued form. It is sometimes called couple stress. The moment stress is the *elastoplastic excitation* with respect to T^a . In other words, at all positions where the

² In the framework of $T(3)$ gauge theory, the dislocation density is identified with the torsion 2-form or object of anholonomy in a Weitzenböck space [1].

³ The non-vanishing of the integral (3.10) has topological reasons, see section 7.

dislocation density is non-vanishing, moment stresses occur. Hence, dislocation theory is a couple or moment stress theory (see also [32]). The physical meaning of the couple stress $H_a = H_{ai} dx^i$ is: the components H^l_i describe twisting moments and the other components describe bending moments [33, 34].

In order to give concrete expressions for the excitation, we have to specify the *constitutive relation* between the field strength T^a and the excitation H_a . We choose a linear constitutive law for an anisotropic material as⁴

$$H_a = \frac{1}{2} \star (\kappa_{aij}{}^{bkl} T_{bkl} dx^i \wedge dx^j) \quad (3.13)$$

where $\kappa_{aij}{}^{bkl}(x)$ are constitutive functions that are characteristic for a crystal with dislocations. These constitutive functions are necessary because the elasticity tensor says nothing about the behaviour in the core of dislocations (plastic region). They have the symmetries

$$\kappa^{aijbkl} = \kappa^{bklaij} = -\kappa^{ajibkl} = -\kappa^{aijblk}. \quad (3.14)$$

For an isotropic material the most general constitutive law is given by

$$H_a = \star \sum_{I=1}^3 a_I {}^{(I)}T_a. \quad (3.15)$$

Here a_1 , a_2 and a_3 are new material constants for a dislocated material. We use the decomposition of the torsion $T^a = {}^{(1)}T^a + {}^{(2)}T^a + {}^{(3)}T^a$ into its $SO(3)$ -irreducible pieces. These three pieces ${}^{(I)}T_a$ are (see also [11])

$${}^{(1)}T^a := T^a - {}^{(2)}T^a - {}^{(3)}T^a \quad (\text{tantor}) \quad (3.16)$$

$${}^{(2)}T^a := \frac{1}{2} \vartheta^a \wedge (e_b \lrcorner T^b) \quad (\text{trator}) \quad (3.17)$$

$${}^{(3)}T_a := \frac{1}{3} e_a \lrcorner (\vartheta^b \wedge T_b) \quad (\text{axitor}). \quad (3.18)$$

The tantor is the torsion corresponding to the Young tableau (2, 1) minus traces. The trator contains the trace terms of the Young tableau (2, 1) and the axitor corresponds to the Young tableau (1, 1, 1) (for group-theoretical notations see, e.g., [35, 36]).

The stress 2-form of dislocations is defined by

$$h_a := \frac{\partial \mathcal{L}_{\text{disl}}}{\partial \vartheta^a} = e_a \lrcorner \mathcal{L}_{\text{disl}} + (e_a \lrcorner T^b) \wedge H_b = \frac{1}{2} [(e_a \lrcorner T^b) \wedge H_b - (e_a \lrcorner H_b) T^b]. \quad (3.19)$$

It is an odd (or axial) vector-valued form. This stress form is called the *Maxwell stress 2-form of dislocations*. Here, h_a is a kind of interaction stress between dislocations which reflects the nonlinearity and universality of interactions of the dislocation theory. It is expressed in terms of dislocation density and moment stress. Thus, h_a describes higher order stresses in the core region. A similar interaction stress of dislocations is discussed in [37, 38].

The definition of the pure *dislocation energy* as the Hamiltonian is given by

$$\mathcal{E}_{\text{core}} := -\mathcal{L}_{\text{disl}} = \frac{1}{2} T^a \wedge H_a. \quad (3.20)$$

More physically, we can interpret $\mathcal{E}_{\text{core}}$ as the static dislocation core energy density.

In order to take boundary conditions into account we use a so-called null Lagrangian [4]:

$$\mathcal{L}_{\text{bg}} = d(\sigma_a^{\text{bg}} \xi^a) = \sigma_a^{\text{bg}} \wedge d\xi^a \longrightarrow \sigma_a^{\text{bg}} \wedge \vartheta^a. \quad (3.21)$$

A null Lagrangian does not change the Euler–Lagrange equations in classical elasticity (force equilibrium) because the background stress σ_a^{bg} is required to satisfy the relation $d\sigma_a^{\text{bg}} = 0$. After minimal replacement in equation (3.21), the Lagrangian \mathcal{L}_{bg} will make contributions to the Euler–Lagrange equations of elastoplasticity (see also [6]).

⁴ A similar constitutive law between moment stress and dislocation density was discussed in [33, 34].

The variation of the total Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{disl}} + \mathcal{L}_{\text{strain}} + \mathcal{L}_{\text{bg}} \tag{3.22}$$

with respect to ξ^a and ϕ^a gives the following field equation in the elastoplastic theory of dislocations in an infinite medium:

$$\frac{\delta \mathcal{L}}{\delta \xi^a} \equiv d\Sigma_a + dh_a = 0 \quad (\text{force equilibrium}) \tag{3.23}$$

$$\frac{\delta \mathcal{L}}{\delta \phi^a} \equiv dH_a - h_a = \hat{\Sigma}_a \quad (\text{moment equilibrium}) \tag{3.24}$$

where the effective stress 2-form, $\hat{\Sigma}_a := \Sigma_a + \sigma_a^{\text{bg}}$, is the driving force stress for the moment stress in equation (3.24). Let us note that, in the framework of MAG, equation (3.24) is the (first) gauge field equation and (3.23) is the matter field equation (see [11]). They are Yang–Mills type field equations of the translational gauge theory.

In order to complete the framework of elastoplastic field theory, we define the elastoplastic forces as field strength \times stress. We introduce the *elastic material force* density by the help of the material stress 2-form via

$$f_a^{\text{el}} = (e_a \rfloor T^b) \wedge \Sigma_b. \tag{3.25}$$

This force contains the contributions due to the eigenstress of dislocations (Peach–Koehler force [39]). The pure dislocation force is given by means of the stress 2-form of dislocations as

$$f_a^{\text{disl}} = (e_a \rfloor T^b) \wedge h_b. \tag{3.26}$$

This force (3.26) characterizes the interaction between dislocations near the dislocation core.

4. What would be a good choice for the moment stress?

In this section we want to discuss different choices for the material constants a_1, a_2 and a_3 . Additionally, we consider the corresponding equations for the moment equilibrium.

For simplicity, we use the weak field approximation (linearization)

$$\xi^a = \delta_i^a x^i + u^a \quad \vartheta^a = (\delta_i^a + \beta_i^a) dx^i \tag{4.1}$$

where u^a is the displacement field and β_i^a is the linear distortion tensor. We note that the dislocation self-interaction stress tensor h_a is of higher order in the Burgers vector. Thus, we neglect it. Here we assume a linear asymmetric (Piola–Kirchhoff) force stress [40]

$$\sigma_a = 2\mu \left(\beta_{(ai)} + \frac{\nu}{1-2\nu} \delta_{ai} \beta_k^k + c_1 \beta_{[ai]} \right) dx^i \tag{4.2}$$

where μ is the shear modulus and ν is Poisson’s ratio. The constant c_1 characterizes the antisymmetric force stress. Then the equation of the moment equilibrium is

$$dH_a = \hat{\Sigma}_a \tag{4.3}$$

with ${}^* \hat{\Sigma}_a = -\hat{\sigma}_a$.

The simplest choice is $a_1 = a_2 = a_3$. It yields the moment stress as

$$H_a = H_{ak} dx^k = \frac{a_1}{2} T_{aij} \epsilon^{ij}_k dx^k = a_1 \alpha_{ai} dx^i \tag{4.4}$$

with $T_{ij}^a = \epsilon_{ijk} \alpha^{ak}$. This is what Edelen did [6] in his gauge theory of dislocations. Accordingly, in this connection we call this choice the ‘Edelen choice’. Eventually, we obtain the field equation (see, e.g., equation (2.6) in [6])

$$\Delta \beta_{ai} - \partial_i \partial^j \beta_{aj} = \frac{1}{a_1} \hat{\sigma}_{ai} \tag{4.5}$$

where Δ denotes the Laplace operator and $\hat{\sigma}_{ai} = \sigma_{ai} - \sigma_{ai}^{\text{bg}}$. Now we use the decomposition $\beta_{ai} = \beta_{(ai)} + \beta_{[ai]}$ and

$$\beta_{(ai)} = \frac{1}{2\mu} \left(\sigma_{(ai)} - \frac{\nu}{1+\nu} \delta_{ai} \sigma_k^k \right) \quad (4.6)$$

$$\beta_{[ai]} = \frac{1}{2\mu c_1} \sigma_{[ai]}. \quad (4.7)$$

We find from Edelen's field equation (4.5) the equations for the symmetric and antisymmetric force stress as

$$\Delta \sigma_{(ai)} - \partial_i \partial^j \sigma_{(aj)} + \frac{\nu}{1+\nu} (\partial_a \partial_i - \delta_{ai} \Delta) \sigma_k^k = \kappa^2 \hat{\sigma}_{(ai)} \quad \kappa^2 = \frac{2\mu}{a_1} \quad (4.8)$$

$$\Delta \sigma_{[ai]} - \partial_i \partial^j \sigma_{[aj]} = c_1 \kappa^2 \hat{\sigma}_{[ai]}. \quad (4.9)$$

Therefore, the Edelen choice in combination with equation (4.2) yields equations for symmetric and antisymmetric force stresses. If we require a symmetrical force stress by setting $c_1 = 0$ in equation (4.2) and use the force equilibrium condition $\partial^i \sigma_{ai} = 0$ as the 'gauge condition', we obtain from equation (4.8) the field equation

$$\Delta \sigma_{(ai)} + \frac{\nu}{1+\nu} (\partial_a \partial_i - \delta_{ai} \Delta) \sigma_k^k = \kappa^2 \hat{\sigma}_{(ai)}. \quad (4.10)$$

Another choice could be $a_1 = a_2 = 0$. Then the axitor:

$${}^{(3)}T_{aij} = \frac{1}{3} (T_{aij} + T_{ija} + T_{jai}) \quad (4.11)$$

defines the moment stress as

$$H_a = \frac{a_3}{6} (T_{aij} + T_{ija} + T_{jai}) \epsilon_k^{ij} dx^k = \frac{a_3}{3} \delta_{ai} \alpha_k^k dx^i. \quad (4.12)$$

By the help of the axitor, the field equation for the distortion field is given by

$$\Delta \beta_{[ai]} - \partial_i \partial^j \beta_{[aj]} + \partial_a \partial^j \beta_{[ij]} = \frac{3}{2a_3} \hat{\sigma}_{ai}. \quad (4.13)$$

If we use equation (4.7) and $\partial^i \sigma_{ai} = 0$, we find the equation

$$\Delta \sigma_{[ai]} = c_1 \tilde{\kappa}^2 \hat{\sigma}_{[ai]} \quad \tilde{\kappa}^2 = \frac{3\mu}{a_3} \quad (4.14)$$

and $\hat{\sigma}_{(ai)} = 0$. It follows from equation (4.14) that this choice of the moment stress requires a pure antisymmetric force stress.

Now we discuss the choice $a_2 = 0$ and $a_3 = -\frac{a_1}{2}$. Then the moment stress is given in terms of the contortion tensor according to

$$H_a = \frac{a_1}{4} (T_{aij} - T_{ija} - T_{jai}) \epsilon_k^{ij} dx^k. \quad (4.15)$$

If we use $T_{ij}^a = \epsilon_{ijk} \alpha^{ak}$, we find the relation between the moment stress tensor H_{ai} and the Nye [41] tensor $\kappa_{ai} = \alpha_{ia} - \frac{1}{2} \delta_{ai} \alpha_k^k$ as

$$H_a = a_1 (\alpha_{ai} - \frac{1}{2} \delta_{ai} \alpha_k^k) dx^i \equiv a_1 \kappa_{ia} dx^i. \quad (4.16)$$

Hence, the moment stress tensor H_{ai} given in this choice is proportional to the transpose of the Nye tensor. The field equation by means of this moment stress is given by

$$\Delta \beta_{(ai)} - \partial_i \partial^j \beta_{(aj)} - \partial_a \partial^j \beta_{[ij]} = \frac{1}{a_1} \hat{\sigma}_{ai}. \quad (4.17)$$

Again, we use equations (4.6) and (4.7) and find the equations for the symmetric and antisymmetric force stress:

$$\Delta\sigma_{(ai)} - \partial_i\partial^j\sigma_{(aj)} + \frac{\nu}{1+\nu}(\partial_a\partial_i - \delta_{ai}\Delta)\sigma_k^k = \kappa^2\hat{\sigma}_{(ai)} \tag{4.18}$$

$$-\partial_a\partial^j\sigma_{[ij]} = c_1\kappa^2\hat{\sigma}_{[ai]}. \tag{4.19}$$

If we put $c_1 = 0$ (symmetrical force stress) and use $\partial^i\sigma_{ai} = 0$, we obtain

$$\Delta\sigma_{(ai)} + \frac{\nu}{1+\nu}(\partial_a\partial_i - \delta_{ai}\Delta)\sigma_k^k = \kappa^2\hat{\sigma}_{(ai)} \tag{4.20}$$

which agrees with equation (4.10).

Another interesting choice seems to be⁵ $a_2 = -a_1$ and $a_3 = -\frac{a_1}{2}$. Then we use the axitor (4.11), the trator

$${}^{(2)}T_{aij} = \frac{1}{2}(\delta_{ai}T_{lj}^l + \delta_{aj}T_{il}^l) \tag{4.21}$$

and the tentor

$${}^{(1)}T_{aij} = T_{aij} - {}^{(2)}T_{aij} - {}^{(3)}T_{aij}. \tag{4.22}$$

Consequently, the moment stress is given by

$$H_a = \frac{a_1}{4} (T_{aij} - T_{ija} - T_{jai} - 2\delta_{ai}T_{lj}^l - 2\delta_{aj}T_{il}^l)\epsilon_k^{ij} dx^k. \tag{4.23}$$

We find the remarkable relationship between the moment stress tensor H_{ai} and the Nye tensor κ_{ai} as

$$H_{ai} = a_1(\alpha_{ia} - \frac{1}{2}\delta_{ai}\alpha_k^k) \equiv a_1\kappa_{ai}. \tag{4.24}$$

Thus, this moment stress tensor H_{ai} is proportional to the Nye tensor. Eventually, we obtain the field equation for the distortion $\beta_{(ai)}$ as

$$a_1\{\Delta\beta_{(ai)} - (\partial_i\partial^k\beta_{(ak)} + \partial_a\partial^k\beta_{(ik)}) + \delta_{ai}\partial^k\partial^l\beta_{(kl)} + \partial_a\partial_i\beta_k^k - \delta_{ai}\Delta\beta_k^k\} = \hat{\sigma}_{(ij)}. \tag{4.25}$$

Because the lhs of equation (4.25) is equivalent to $(\text{inc}\beta)_{(ij)}$, equation (4.25) is the proper gauge theoretical formulation of Kröner's incompatibility equation (see, e.g., equation (II.21) in [43]). Kröner's incompatibility tensor is replaced by the effective stress tensor on the rhs of this equation. Additionally, we rewrite the field equation in terms of the force stress as

$$\Delta\sigma_{(ai)} + \frac{1}{1+\nu}(\partial_a\partial_i - \delta_{ai}\Delta)\sigma_k^k = \kappa^2\hat{\sigma}_{(ai)}. \tag{4.26}$$

For $\hat{\sigma}_{(ai)} = 0$ this equation is the Beltrami equation. Let us emphasize that the factor $1/(1+\nu)$ in equation (4.26) differs from the factor $\nu/(1+\nu)$ in equations (4.10) and (4.20). Another interesting point is that, by means of the 'Einstein choice', the force stress is symmetric in a quite natural way. We do not have to make any assumption with respect to the material constant c_1 . Therefore, in order to investigate dislocations with symmetric force stress the 'Einstein choice' of the constants a_1 , a_2 , and a_3 is favourable and will be used in the following.

⁵ This choice of parameters is called the 'Einstein choice' and can be obtained from the condition that the gauge Lagrangian $\mathcal{L}_{\text{disl}}$ has to be invariant under local $SO(3)$ -transformations in order to obtain the teleparallel version of the Hilbert-Einstein Lagrangian in three dimensions [42].

5. A straight screw dislocation in linear approximation

5.1. Field equation and stress field

For simplicity, we consider a straight screw dislocation in an isotropic and incompressible medium ($B = 1$) in linear approximation. For a straight screw dislocation, the Burgers vector and the dislocation line are parallel. In this case, the problem has *cylindrical* symmetry.

In the framework of the elastoplastic field theory, we require that the modified stress field of a screw dislocation has the following properties: (i) the stress field should have no singularity at $r = 0$, and (ii) the far field stress ought to be the stress field of a Volterra dislocation σ_{ij}^{bg} which satisfies the condition $\partial^j \sigma_{ij}^{\text{bg}} = 0$. Thus the condition (ii) is a boundary condition for the stress of a dislocation in the field theory of elastoplasticity. We choose the dislocation line and Burgers vector in the z axis of a Cartesian coordinate system. Then the background stress is given by the elastic stress of a Volterra screw dislocation [44]:

$$\sigma_{xz}^{\text{bg}} = \sigma_{zx}^{\text{bg}} = -\frac{\mu b}{2\pi} \frac{y}{r^2} \quad \sigma_{yz}^{\text{bg}} = \sigma_{zy}^{\text{bg}} = \frac{\mu b}{2\pi} \frac{x}{r^2} \quad (5.1)$$

where $r^2 = x^2 + y^2$. Obviously, these stress fields are singular at the dislocation line.

We turn to equation (4.26) and put $\sigma_k^k = 0$. Then the field equation for the force stress of a linear screw dislocation is given by the following inhomogeneous Helmholtz equation:

$$(1 - \kappa^{-2} \Delta) \sigma_{(ij)} = \sigma_{(ij)}^{\text{bg}}. \quad (5.2)$$

Thus, for the elastic strain fields,

$$(1 - \kappa^{-2} \Delta) E_{ij} = E_{ij}^{\text{bg}}. \quad (5.3)$$

If we put $\sigma_k^k = 0$ in equations (4.10) and (4.20), we also obtain equation (5.2).

Now we seek a cylindrically symmetric (string-like) solution of a screw dislocation. One finds the solution for the distortion field

$$\beta_{zx} = -\frac{y}{r^2} \left(\frac{b}{2\pi} + C_1 r K_1(\kappa r) \right) \quad \beta_{zy} = \frac{x}{r^2} \left(\frac{b}{2\pi} + C_1 r K_1(\kappa r) \right) \quad (5.4)$$

where K_1 is the modified Bessel function of the second kind of order one. This solution is similar to the potential of a magnetic vortex (Abrikosov–Nielsen–Olesen string) for a constant Higgs field [13, 14]. The constant of integration C_1 is determined from the condition that the distortion β_{zx} and β_{zy} vanish at $r = 0$ ($\kappa r \ll 1$ with $C_1 K_1(\kappa r) \approx C_1 \frac{1}{\kappa r}$) as

$$C_1 = -\frac{b\kappa}{2\pi}. \quad (5.5)$$

With (5.5) we obtain for the distortion of a screw dislocation

$$\beta^z = \frac{b}{2\pi r^2} (1 - \kappa r K_1(\kappa r)) (x \, dy - y \, dx). \quad (5.6)$$

The distortion 1-form can be expressed in cylindrical coordinates as

$$\beta^z = \frac{b}{2\pi} (1 - \kappa r K_1(\kappa r)) \, d\varphi. \quad (5.7)$$

The deformation (5.7) around the screw dislocation is a pure shear. The effective Burgers vector is given by

$$b^z(r) = \oint_{\gamma} \beta^z = b \{1 - \kappa r K_1(\kappa r)\}. \quad (5.8)$$

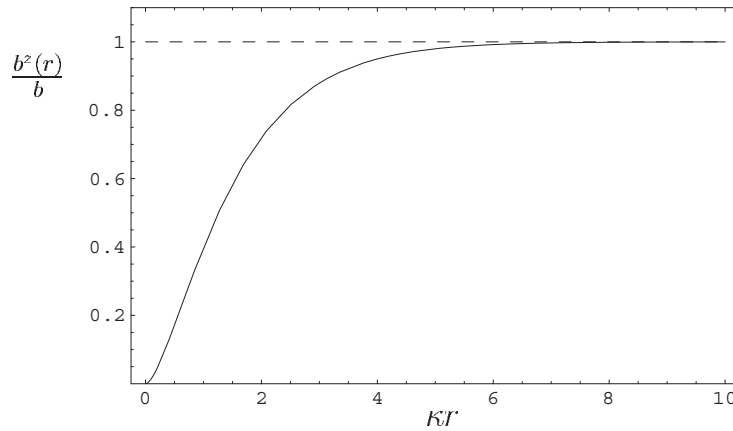


Figure 1. Effective Burgers vector $b^z(r)/b$ (full curve).

This effective Burgers vector $b^z(r)$ differs from the constant Burgers vector b in the region from $r = 0$ up to $r = 6/\kappa$ ($\{1 - \kappa r K_1(\kappa r)\}|_{\kappa r=6} = 0.992$) because the distortion field is modified due to the moment stress (see figure 1).

Let us now rewrite the distortion (5.7) as follows:

$$\begin{aligned} \beta^z &= \frac{b}{2\pi} (d\{(1 - \kappa r K_1(\kappa r))\varphi\} - \varphi \kappa^2 r K_0(\kappa r) dr) \\ &\equiv du^z + \phi^z \end{aligned} \tag{5.9}$$

where $\varphi = \arctan(y/x)$. We can interpret the field

$$\phi^z = -\frac{b\kappa^2}{2\pi} \varphi r K_0(\kappa r) dr \tag{5.10}$$

as the proper incompatible part (negative plastic distortion) of the distortion due to $d\phi^z \neq 0$. It vanishes at $r = 0$ and $r \rightarrow \infty$ (see figure 2). The compatible part of equation (5.9) is

$$u^z = \frac{b}{2\pi} (1 - \kappa r K_1(\kappa r))\varphi \tag{5.11}$$

and is a modified displacement field (see figure 3). This u^z is multivalued and has no singularity. The asymptotic form of u^z is the classical displacement function $\frac{b}{2\pi}\varphi$ and it vanishes at $r = 0$. Moreover, we observe that the classical displacement function is only a kind of *phase-only* approximation and is not valid in the dislocation core analogous to the Higgs field in superconductors or in string theory. Obviously, the displacement field u^z plays the role of a *Higgs field* in the elastoplastic theory. The line $u^z = 0$ is surrounded by a tube of radius $\approx r_c$, the dislocation core, within which $b^z(r)$ is suppressed from its constant value b . From this point of view, we may identify the length

$$r_c \simeq 6/\kappa \tag{5.12}$$

as the *dislocation core radius*. Outside the core radius the classical elasticity describes dislocations very well. Thus, the core radius is the inner cut-off radius of classical elasticity where linear elasticity theory should apply. In this framework, equation (5.11) describes the ‘atomic’ arrangement in the core region (hopefully to a good approximation). The local atomic configuration inside the core region is fundamentally different from that of the defect-free parts of the crystal. Therefore, a dislocation is a defect breaking locally the translation invariance in the core region.

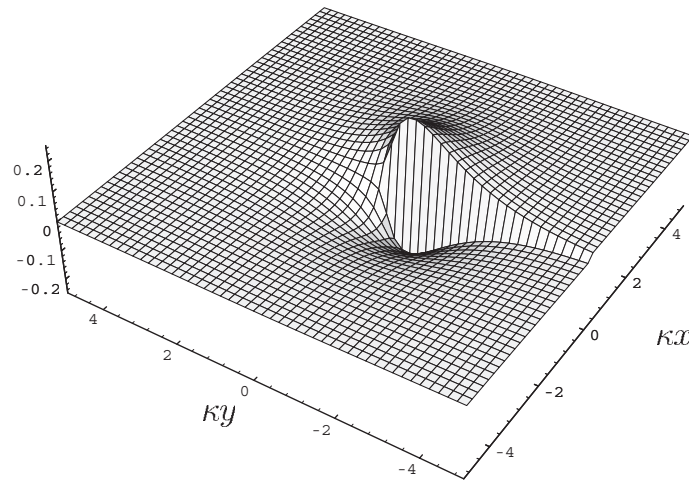


Figure 2. Incompatible plastic distortion $-\phi^z/b\kappa$.

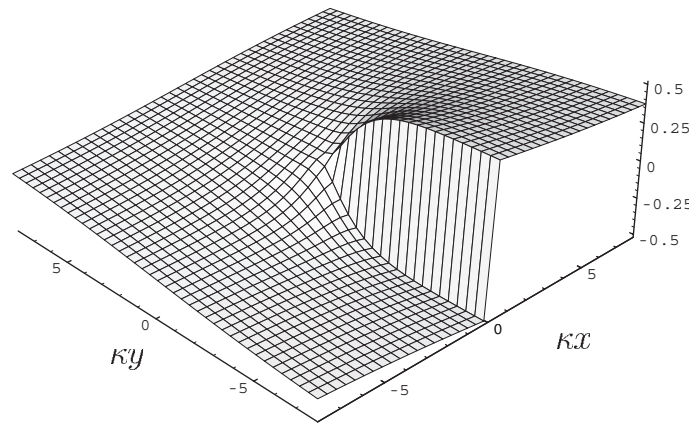


Figure 3. Displacement field u^z/b .

Let us mention that Edelen and Kadić [2, 3] imposed the conditions $\xi^a = x^a$ (no elastic displacements) and $\sigma_a^{\text{bg}} = 0$ (no background stress) in their investigation of dislocation-type solutions. Accordingly, they used the translational part of the affine connection instead of the generalized affine connection. That was the reason why they obtained short-reaching solutions which are ‘unphysical’.

After all these considerations, we may identify the mapping function from the defect-free to the distorted configuration according to

$$\xi^z = z + \frac{b}{2\pi} (1 - \kappa r K_1(\kappa r)) \varphi. \quad (5.13)$$

The corresponding anholonomic coframe of the inner geometry is given by

$$\vartheta^r = dr \quad \vartheta^\varphi = r d\varphi \quad \vartheta^z \equiv dz + \beta^z = dz + \frac{b}{2\pi} (1 - \kappa r K_1(\kappa r)) d\varphi. \quad (5.14)$$

This coframe has a helical structure and no artificial singularity.

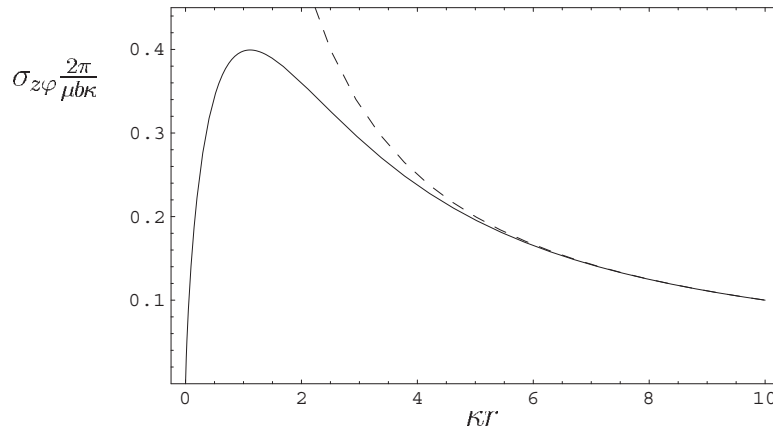


Figure 4. Force stress of a screw dislocation $\sigma_{z\varphi}(2\pi/\mu b\kappa)$ (full curve) and classical $1/r$ stress (broken curve).

The force stress of a screw dislocation is given by

$$\sigma_{z\varphi} = \sigma_{\varphi z} = \frac{\mu b}{2\pi r} (1 - \kappa r K_1(\kappa r)). \tag{5.15}$$

This eigenstress of a screw dislocation is modified near the dislocation core (up to $6/\kappa$) and decays like r^{-1} for large r . It does not possess singularity at $r = 0$. The eigenstress has a maximum at $r \simeq 1.1\kappa^{-1}$ (see figure 4):

$$\sigma_{z\varphi}^{\max} \simeq 0.399 \frac{\mu b}{2\pi} \kappa. \tag{5.16}$$

Consequently, it is not true that the eigenstress of a screw dislocation decays exponentially with distance r far from the core.

Let us mention that the modified stress field (5.15) agrees with Eringen’s stress field [45,46] which is calculated in the framework of *nonlocal elasticity*. Additionally, it is interesting to note that the stress field (5.15) is the same as the one obtained by Gutkin and Aifantis [47,48] in their version of *gradient elasticity*.

5.2. Torsion, moment stress, energy and force of screw dislocations

Let us now apply the gauge potential of a screw dislocation in order to calculate the torsion, moment stress, elastoplastic energy and dislocation core energy. Additionally, we compute the modified Peach–Koehler force between two screw dislocations.

The nonvanishing components of torsion are now calculated by means of the dislocation potential similar to a magnetic vortex in cylindrical coordinates:

$$T^z = -\frac{b\kappa}{2\pi} \frac{\partial}{\partial r} (r K_1(\kappa r)) \, dr \wedge d\varphi = \frac{b\kappa^2}{2\pi} r K_0(\kappa r) \, dr \wedge d\varphi \tag{5.17}$$

and in Cartesian coordinates as

$$T^z = \frac{b\kappa^2}{2\pi} K_0(\kappa r) \, dx \wedge dy \tag{5.18}$$

where K_0 is the modified Bessel function of the second kind of order zero (see figure 5). Let us note that this elastoplastic field strength (torsion) is analogous to the magnetic field strength of a magnetic vortex. Of course, the torsion T^z fulfils the Bianchi identity $dT^z = 0$.

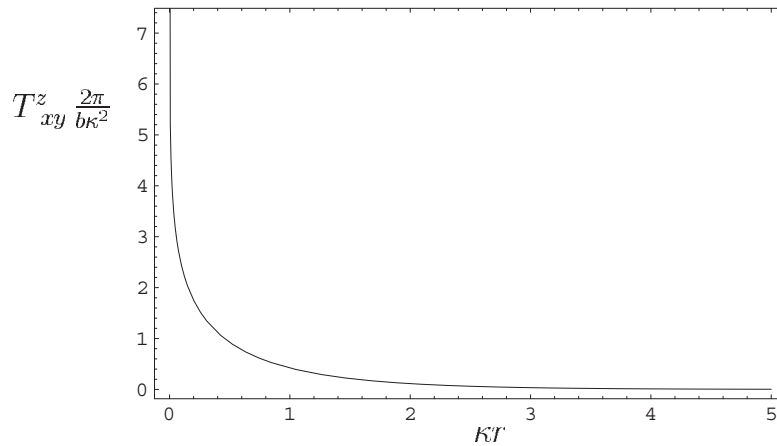


Figure 5. Dislocation density $T_{xy}^z(2\pi/b\kappa^2)$.

Note that $T_{r\phi}^z \approx -\frac{b\kappa^2}{2\pi}(\ln \frac{\kappa r}{2} + \gamma)$ for $r \ll \kappa^{-1}$ (near field) and $T_{r\phi}^z \approx \frac{b\kappa^2}{2\sqrt{2\pi\kappa r}} \exp(-\kappa r)$ for $r \gg \kappa^{-1}$ (far field). Thus the far field of torsion decreases exponentially with r , with the characteristic length κ^{-1} . When $\kappa^{-1} \rightarrow 0$ in (5.17) we obtain the Dirac delta function as torsion and dislocation density, respectively, so that the ‘classical’ dislocation density is reverted in this limit. Additionally, we observe that the dislocation density (5.17) agrees with Eringen’s two-dimensional nonlocal modulus which was obtained by matching the phonon dispersion curves [45]. Now, we define the *plastic penetration depth*:

$$R_c := \frac{1}{\kappa} = \sqrt{\frac{a_1}{2\mu}} \tag{5.19}$$

as the region over which the torsion is appreciably different from zero and the torsional flux is confined within this region. Then R_c measures the proper plastic region where $\phi^a \neq 0$. Moreover, the new constant a_1 is determined through R_c and κ , respectively. The maximum of the stress is in the Peierls–Nabarro model [49, 50] given as $\mu/2\pi$ with $b = a$, where a is the lattice parameter. If we compare our result (5.16) with this maximum of the stress field, it is possible to determine the unknown factor κ as

$$\kappa^{-1} \simeq 0.399a. \tag{5.20}$$

Therefore, by means of (5.20), the typical material constant for a screw dislocation is given by

$$a_1 \simeq 2\mu(0.399a)^2. \tag{5.21}$$

Let us mention that Eringen has already obtained an analogous result for κ^{-1} in his nonlocal elasticity theory [45]. He pointed out that the choice of $\kappa^{-1} \simeq 0.399a$ excellently matches with experimental atomic dispersion curves. For this value of κ^{-1} the core radius is given as $r_c \simeq 2.4a$. The stress field has its maximum $\mu b/2\pi$ at $r \simeq 0.44a$. Gutkin and Aifantis [47, 48] have used another choice of the factor κ as $\kappa^{-1} \simeq 0.25a$ so that the core radius is $r_c \simeq 1.50a$. Thus, the factor κ determines the position and the magnitude of the stress and strain maxima. Finally, the factor κ should be fitted by comparing predictions of the theory with experimental results and computer simulations.

The presence of dislocations gives rise to a localized moment stress. This moment stress 1-form is given by the help of equation (4.24) as

$$H_z = \frac{\mu b}{2\pi} K_0(\kappa r) dz \quad H_x = -\frac{\mu b}{2\pi} K_0(\kappa r) dx \quad H_y = -\frac{\mu b}{2\pi} K_0(\kappa r) dy. \tag{5.22}$$

These moment stresses mean physically twisting moments in the dislocation core region. We find for the Nye tensor

$$\kappa_{zz} = \frac{b\kappa^2}{4\pi} K_0(\kappa r) \quad \kappa_{xx} = -\frac{b\kappa^2}{4\pi} K_0(\kappa r) \quad \kappa_{yy} = -\frac{b\kappa^2}{4\pi} K_0(\kappa r). \quad (5.23)$$

The Nye tensor and the moment stress are appreciably different from zero in the region $r \leq R_c$.

Now we are able to calculate the strain and the core energy in this framework. The *stored strain energy* of a screw dislocation per unit length is given by

$$\begin{aligned} E_{\text{strain}} &= \frac{\mu b^2}{4\pi} \int_0^R dr r \left(\frac{1}{r} - \kappa K_1(\kappa r) \right)^2 \\ &= \frac{\mu b^2}{4\pi} \left\{ \ln(r) + 2K_0(\kappa r) + \frac{\kappa^2 r^2}{2} (K_1(\kappa r)^2 - K_0(\kappa r)K_2(\kappa r)) \right\} \Big|_0^R \end{aligned} \quad (5.24)$$

where R is the outer cut-off radius. We use the limiting expressions for $r \rightarrow 0$:

$$K_0(\kappa r) \approx -\gamma - \ln \frac{\kappa r}{2} \quad K_1(\kappa r) \approx \frac{1}{\kappa r} \quad K_2(\kappa r) \approx -\frac{1}{2} + \frac{2}{(\kappa r)^2} \quad (5.25)$$

where $\gamma = 0.57721566\dots$ is the Euler constant, and for $r \rightarrow \infty$:

$$K_n(\kappa r) \approx \frac{\sqrt{\pi} \exp(-\kappa r)}{\sqrt{2\kappa r}}. \quad (5.26)$$

The final result for the strain energy is

$$E_{\text{strain}} = \frac{\mu b^2}{4\pi} \left\{ \ln \frac{\kappa R}{2} + \gamma - \frac{1}{2} \right\}. \quad (5.27)$$

Thus we obtain a strain energy density which is not singular at the dislocation line. The *dislocation core energy* per unit length is

$$\begin{aligned} E_{\text{core}} &= \frac{\mu b^2 \kappa^2}{4\pi} \int_0^\infty dr r K_0(\kappa r)^2 \\ &= \frac{\mu b^2 \kappa^2}{8\pi} r^2 \{ K_0(\kappa r)^2 - K_1(\kappa r)^2 \} \Big|_0^\infty \\ &= \frac{\mu b^2}{8\pi} \end{aligned} \quad (5.28)$$

which agrees, up to a factor of 2, with the core or misfit energy that is calculated for the screw dislocation in the Peierls–Nabarro model [44]. Finally, we obtain for the total energy (per unit length) of a screw dislocation

$$E_{\text{screw}} = \frac{\mu b^2}{4\pi} \left\{ \ln \frac{\kappa R}{2} + \gamma \right\}. \quad (5.29)$$

Due to the fact that the Burgers vector is quantized (see section 7), the core and strain energy of a dislocation is also quantized.

Now we recover the Peach–Koehler force from equation (3.25) as

$$\begin{aligned} f_a^{\text{el}} &\equiv -f_a^{\text{PK}} \\ &= -\partial_{[a} \beta_{j]}^b \sigma_b^l \epsilon_{lmn} dx^j \wedge dx^m \wedge dx^n. \end{aligned} \quad (5.30)$$

For straight screw dislocations the Peach–Koehler force is given in the framework of linear dislocation gauge theory as a radial force density

$$f_r^{\text{PK}} = \partial_r (\beta^{z\varphi} \sigma_{z\varphi}) \eta. \quad (5.31)$$

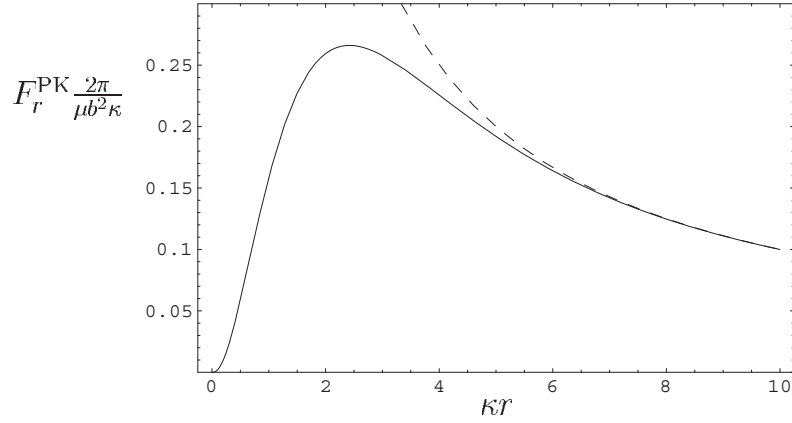


Figure 6. Peach–Koehler force $F_r^{\text{PK}}(2\pi/\mu b^2 \kappa)$ (full curve) and classical $1/r$ force (broken curve).

We obtain for the force per unit length acting on one screw dislocation in the stress field due to the other screw dislocation from equation (5.24)

$$\begin{aligned} F_r^{\text{PK}} &= 2\partial_r E_{\text{strain}} \\ &= \frac{\mu b^2}{2\pi r} (1 - 2\kappa r K_1(\kappa r) + \kappa^2 r^2 K_1(\kappa r)^2). \end{aligned} \quad (5.32)$$

Here E_{strain} is the interaction strain energy between the two parallel screw dislocations. We see that the modified Peach–Koehler force is attractive for screw dislocations of opposite sign, and repulsive for dislocations of the same sign and is far-reaching. This Peach–Koehler force is modified near the dislocation core (up to $6\kappa^{-1}$) and decays like r^{-1} for large r . It does not possess any singularity at $r = 0$. The modified Peach–Koehler force has a maximum at $r \simeq 2.42\kappa^{-1}$ (see figure 6).

6. Dislocation theory as three-dimensional gravity

In the preceding section, we have described the dislocation theory as a Weitzenböck space (teleparallelism) with nontrivial torsion T^a . An alternative description of dislocation theory is to consider the body manifold as a Riemann space with Christoffel symbols as a connection and nontrivial Riemannian curvature. In the last case the dislocation theory is equivalent to three-dimensional gravity [42]. In this picture, the Cauchy–Green strain tensor is the gravitational field which describes the deformation of the manifold from the undeformed one.

The Levi-Civita connection (Christoffel symbol) $\tilde{\omega}_{ab}$, corresponding to the metric (Cauchy–Green tensor) $G = \delta_{ab} \vartheta^a \otimes \vartheta^b$, can be derived from the contortion 1-form τ_{ab} by means of the teleparallel condition:

$$\tilde{\omega}_{ab} - \tau_{ab} = \omega_{ab} \equiv 0 \quad \implies \quad \tilde{\omega}_{ab} = \tau_{ab}. \quad (6.1)$$

Then the Levi-Civita connection is given by

$$\tilde{\omega}_{ab} = \frac{1}{2}(-T_{abi} - T_{bia} + T_{iab}) dx^i. \quad (6.2)$$

The corresponding Riemannian curvature 2-form is

$$\tilde{R}_{ab} = \frac{1}{2} \tilde{R}_{abij} dx^i \wedge dx^j = d\tilde{\omega}_{ab} + \tilde{\omega}_{ac} \wedge \tilde{\omega}_{cb}. \quad (6.3)$$

Eventually, we get the corresponding field equation by using the three-dimensional Hilbert–Einstein Lagrangian:

$$\mathcal{L}_{\text{HE}} = -\frac{1}{2\ell} \tilde{R}^{ab} \wedge \eta_{ab} \tag{6.4}$$

instead of the teleparallel Lagrangian $\mathcal{L}_{\text{disl}}$ in the ‘Einstein choice’ of the three parameters ($a_1 = 1, a_2 = -1, a_3 = -\frac{1}{2}$) as shown in [1, 5] by help of the following remarkable identity⁶ (see, e.g., [42]):

$$-R^{ab} \wedge \eta_{ab} + \tilde{R}^{ab} \wedge \eta_{ab} - T^a \wedge \star \left({}^{(1)}T_a - {}^{(2)}T_a - \frac{1}{2} {}^{(3)}T_a \right) \equiv 2d(\vartheta^a \wedge \star T_a). \tag{6.5}$$

Consequently, in a Weitzenböck space with vanishing Riemann–Cartan curvature, i.e. $R^{ab} = 0$, the Lagrangian $\mathcal{L}_{\text{disl}}$ is, up to a boundary term, equivalent to the Hilbert–Einstein Lagrangian \mathcal{L}_{HE} in three dimensions⁷. After variation with respect to ϑ^a , one recovers an Einstein-type field equation

$$\tilde{G}_a \equiv \frac{1}{2} \eta_{abc} \tilde{R}^{bc} = \ell \hat{\Sigma}_a. \tag{6.6}$$

Here ℓ is the coupling constant of ‘dislocation gravity’.

Let us analyse the Riemannian geometry caused by a screw dislocation by using the torsion (5.18) and the elastoplastic stress tensor. The nonvanishing components of the Einstein tensor are

$$\begin{aligned} \tilde{G}_x &= -\frac{b\kappa^3}{4\pi r} K_1(\kappa r) y \, dx \wedge dy, \\ \tilde{G}_y &= \frac{b\kappa^3}{4\pi r} K_1(\kappa r) x \, dx \wedge dy \\ \tilde{G}_z &= -\frac{b\kappa^3}{4\pi r} K_1(\kappa r) (x \, dx \wedge dz + y \, dy \wedge dz). \end{aligned} \tag{6.7}$$

The source of the Einstein tensor is the following effective stress tensor:

$$\begin{aligned} \hat{\Sigma}_x &= -\frac{\mu b\kappa}{2\pi r} K_1(\kappa r) y \, dx \wedge dy \\ \hat{\Sigma}_y &= \frac{\mu b\kappa}{2\pi r} K_1(\kappa r) x \, dx \wedge dy \\ \hat{\Sigma}_z &= -\frac{\mu b\kappa}{2\pi r} K_1(\kappa r) (x \, dx \wedge dz + y \, dy \wedge dz) \end{aligned} \tag{6.8}$$

which is the eigenstress of a screw dislocation without the ‘classical’ displacement field (Higgs field) $u^z = \frac{b}{2\pi} \varphi$. We see that the field u^z gives no contribution to the Einstein tensor (6.7) and to the stress tensor (6.8). All this looks like general relativity where we are internal observers. One may imagine that an external observer is able to deform our universe from outside, but this deformation would be compatible and therefore not felt by us as internal observers [51]. Is, perhaps, this observation a hint why we may use the Cartan or affine connection instead of the generalized affine connection in gravity? However, the ‘Higgs field’ u^a should play a physical role in gravity, too, e.g. as a nontrivial vacuum.

Obviously, the dislocation acts as the source of an incompatible ‘gravitational’ distortion field and is its own source. Additionally, we can say that a screw dislocation is a topological string with cylindrical symmetry in three-dimensional gravity. Perhaps dislocations in crystals provide a better experimental field for testing gravity models with cosmic strings (see also [52]). Let us note that the interesting analogy between vortices in superfluids and spinning cosmic strings is discussed in [53].

⁶ Note that $a_2 = -2$ in [1] is unfortunately a misprint.

⁷ The gauge theoretical description of dislocation theory based on the Lagrangian \mathcal{L}_{HE} in combination with an elastic Lagrangian is also proposed by Malyshev [5] and is equivalent to the teleparallel formulation in this paper.

Now we determine the ‘gravitational’ constant ℓ for a screw dislocation. After substituting of (6.7) and (6.8) in (6.6), we observe

$$\ell = \frac{1}{a_1}. \quad (6.9)$$

We are discussing the material tungsten (W) because it is nearly isotropic. With the lattice constant, $a = 3.16 \times 10^{-10}$ m, and the shear modulus, $\mu = 1.61 \times 10^{11}$ N m⁻², we obtain with (5.21)

$$\ell \simeq 1.95 \times 10^8 \text{ N}^{-1}. \quad (6.10)$$

Remarkable, the ‘gravitational’ constant in dislocation theory is much bigger than the Einstein gravitational constant $\ell_E = 2.08 \times 10^{-43}$ N⁻¹. But this is not surprising because the Planck length, $a_{\text{Pl}} = 1.62 \times 10^{-35}$ m, is much smaller than the lattice constant of the crystal. Because the coupling constant is quadratic in the specific length, the difference between the gravitational constant in dislocation theory and Einstein gravity should be a factor of 10^{50} —and this is what we get.

From the quantum mechanical point of view, a crystal has a lattice structure. We are able to measure the lattice constants by means of x-ray diffraction or transmission electron microscopy and we observe that a crystal is not a continuum. Nevertheless, if the energy of the particles is so small that the lattice structure cannot be resolved, then the differential geometric specification provides an effective description of the *continuized* crystal (see also [54]). A continuized crystal is the result of a limiting process in which the lattice parameter is more and more reduced such that the mass density and the crystallographic directions remain unchanged [55]. Therefore, the dislocation theory as three-dimensional gravity by means of the Einstein field equations is a low energy description, like the Einstein theory of gravity. But in gravity we do not have a ‘microscope’ in order to observe the typical length and lattice parameter, respectively.

7. Some topological remarks about dislocations in crystals

Let us now discuss some topological properties of dislocations in crystals (see [56–58]). Due to the Burgers circuit (3.10), a dislocation is a topological line defect and the body manifold \mathcal{M}^3 is not simply connected (that is, if \mathcal{M}^3 contains incontractible loops). In general, line defects, e.g. dislocation, vortices and cosmic strings, are described by the first homotopy group π_1 .

The three-dimensional crystal is described by the discrete translations in three dimensions (Bravais lattice vectors). The isotropy group of the crystal is the group of discrete translations \mathbb{Z}^3 . If we identify points differing by a primitive lattice vector, we see that the one-dimensional translation group $T(1)$ is mapped to the group $U(1)$ and the one-dimensional sphere S^1 , respectively. After this periodic boundary condition the corresponding space is identified with the three-dimensional torus $T^3 \cong T(3)/\mathbb{Z}^3 = S^1 \times S^1 \times S^1$ and the continuous translation group is broken to the discrete translation group: $T(3) \rightarrow \mathbb{Z}^3$. The type of defect depends on the topology of T^3 . The first fundamental group of the three-dimensional torus as the coset space is $\pi_1(T^3) = \mathbb{Z}^3$. Thus, from the topological point of view, the dislocations are characterized by a Bravais lattice vector $\mathbf{b} = u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3$, called the Burgers vector. Here $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ are the primitive lattice vectors and $(u, v, w) \in \pi_1(T^3)$. Thus the Burgers vector is quantized. Due to the non-vanishing of the first homotopy group $\pi_1(T^3) = \mathbb{Z}^3$, the underlying fibre bundle is topologically nontrivial.

We have seen that the dislocations are topological defects similar to vortices, where the magnetic flux is quantized. In general, topological defects are known as topological charges in gauge theories. The quantized abelian topological charge of dislocations is the Burgers vector that is the torsional flux. Hence, a dislocation is a kind of a torsion vortex in crystals.

Dislocations have (pseudo)-particle-like properties. For example, they may annihilate with their ‘anti-particles’, i.e. dislocations of opposite Burgers vector. Let us remark that Seeger [59] has already considered dislocations as solitons in crystals, namely as global solitons or kinks which are a solution of the Enneper or sine-Gordon equation.

Some topological remarks about static dislocations have been discussed by Gairola [60]. But he has not clarified the nature of the gauge field, which we have identified with the dislocation gauge potential in the framework of translation gauge theory [1].

Dislocations in crystals can be described by means of the Burgers vector and the direction of the dislocation line s . One usually distinguishes between screw ($b \parallel s$) and edge ($b \perp s$) dislocations. But from the topological point of view both these types are equivalent.

8. Conclusions

We have proposed a static theory of dislocations with moment stress which represents the specific response to dislocation distributions in an anisotropic or isotropic elastoplastic material as a three-dimensional translation gauge theory. We have explicitly seen that a physical field theory of dislocations has to contain the notion of moment stress. Hence, dislocation theory is a couple or moment stress theory. In this theory of dislocations the force stress vanishes, except at the positions of the dislocations, where it gives rise to a localized moment stress. Obviously, the size of this moment stress cannot be calculated from classical elasticity theory. Thus, a field theory of dislocations without moment stress is obsolete.

In our theory we have used the framework of MAG and the analogy between the dislocation theory and Maxwell’s theory. In order to obtain a field theory, we have used the concepts of field strength, excitation and constitutive law analogous to the electromagnetic field theory. All elastoplastic field quantities can be described by \mathbb{R}^3 -valued exterior differential forms. The elastoplastic field strength is an even (or polar) differential form and the moment and the force stress are odd (or axial) forms. We have shown that the elastoplastic excitation with respect to dislocation density is necessary for a realistic physical dislocation theory. As a constitutive relation between dislocation density and moment stress we have discussed linear laws for isotropic and anisotropic materials. For isotropic materials we used the teleparallel Lagrangian, which is equivalent to the Hilbert–Einstein Lagrangian, as the dislocation gauge Lagrangian. In this case, the constitutive relation between the dislocation density and the moment stress is compatible with the constitutive law between the strain and the symmetrical force stress. Moreover, we have proven that the moment stress in the ‘Einstein choice’ is proportional to the Nye tensor. A new material constant a_1 enters in the constitutive relation between the dislocation density and the moment stress. It defines a new internal length scale κ^{-1} .

Additionally, we have demonstrated how to fit the excitations into the Maxwell type field equations, in contrast to [61] who claimed that there are no analogues to the second pair of Maxwell equations in dislocation theory. A static dislocation theory is analogous to the magnetostatics. We have used the analogy between fields which have the same field theoretical meaning (differential forms of the same degree). Therefore, from the field theoretical point of view, this analogy is more straightforward than the analogy used by Kröner [62]. Moreover, we have pointed out the analogy between a magnetic (Abrikosov–Nielsen–Olesen) vortex and a screw dislocation in a crystal. Consequently, a dislocation is a translational vortex or string. A review of the corresponding magnetic and dislocation quantities is given in table 1.

Additionally, we discussed the dislocation theory as a gravity theory in three dimensions. We pointed out some similarities between dislocations and cosmic strings.

In the classical theory of dislocations one usually claims that the dislocation core cannot be described in linear approximation because of the singularity of the stress field at $r = 0$ and

Table 1. The correspondence between a magnetic vortex and a screw dislocation.

| | |
|------------------------------------------------------------------|-------------------------------------------------------------|
| B —magnetic field strength | T^a —dislocation density |
| H —magnetic excitation | H_a —moment stress |
| A —magnetic potential | ϑ^a —incompatible distortion |
| f —Higgs field | ξ^a —mapping function |
| $B = dA$, $A = A' + df$ | $T^a = d\vartheta^a$, $\vartheta^a = \phi^a + d\xi^a$ |
| Coulomb gauge: | Coulomb gauge: |
| $d^*A = 0$ | $d^*\vartheta^a = 0$ |
| Magnetic flux (n -winding number): | Burgers vector: |
| $\Phi_0 = n\pi\hbar c/e_0$ | b^a must be a lattice vector |
| $\oint_\gamma A = \Phi_0$ | $\oint_\gamma \vartheta^a = b^a$ |
| j —electric current | $\Sigma_a^T = \hat{\Sigma}_a + h_a$ —force stress |
| Gauge potential of a magnetic string ($n = 1$): | Distortion of a screw dislocation: |
| $A = \Phi_0/(2\pi)(1 - \lambda r K_1(\lambda r))d\varphi$ | $\beta^z = b/(2\pi)(1 - \kappa r K_1(\kappa r))d\varphi$ |
| Field strength of a magnetic string ($n = 1$): | Torsion of a screw dislocation: |
| $B = \Phi_0\lambda^2/(2\pi) r K_0(\lambda r) dr \wedge d\varphi$ | $T^z = b\kappa^2/(2\pi) r K_0(\kappa r) dr \wedge d\varphi$ |
| Magnetic field closed: | Dislocation density closed: |
| $dB = 0$ | $dT^a = 0$ |
| (Static) Oersted–Ampère law: | Moment stress equilibrium: |
| $dH = j$ | $dH_a = \Sigma_a^T$ |
| Continuity equation: | Force stress equilibrium: |
| $dj = 0$ | $d\Sigma_a^T = 0$ |
| Constitutive law: | Constitutive law: |
| $H = H(B)$ | $H_a = H_a(T^b)$ |
| Magnetic energy density: | Energy density of dislocations: |
| $\mathcal{E}_{em} = \frac{1}{2}B \wedge H$ | $\mathcal{E}_{disl} = \frac{1}{2}T^a \wedge H_a$ |

that one has to use the nonlinear elasticity near the core. The reason for this assumption is that the classical theory of dislocations does not use a constitutive law between dislocation density as the elastoplastic field strength and the moment stress as the elastoplastic excitation in a field theoretical way. In the elastoplastic field theory, it is possible to describe the core region even in linear approximation very well.

Two characteristic distances appear naturally in this approach: the dislocation core radius $r_c \simeq 6\kappa^{-1}$ and the plastic penetration depth $R_c \simeq \kappa^{-1}$ which may be viewed as the radius of the region over which the dislocation density (torsion), the Nye tensor and the moment stress are appreciably different from zero. We found in this theory of dislocations with moment stress that the near stress field for a screw dislocation is modified up to $r_c \simeq 6\kappa^{-1}$ (core radius) and the far field is in agreement with the classical stress field. Thus the translation gauge theory of dislocations removes the artificial singularity at the core of classical dislocation theory. It gives the correct results of the elasticity theory for a screw dislocation and the modification in the core region due to the moment stress. We have discussed the choice of the coupling constant between the dislocation density and the moment stress as $a_1 \simeq 2\mu(0.399a)^2$ for a screw dislocation. Accordingly, we found that the Burgers vector is also modified in the region from $0 \leq r \lesssim 2.4a$. Moreover, we calculated the dislocation density, the moment stress and the elastoplastic energy of a screw dislocation. We have shown that this translational gauge model is useful in determining the width of a screw dislocation and in estimating the core energy of a screw dislocation similar to the Peierls–Nabarro model.

Last, but by no means least, we have seen that the translational gauge theory of dislocations is a field theory where the torsion and the translational part of the generalized affine connection play a physical role.

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